Optimal Extraction of Structural Characteristics from Response Measurements

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When random input data are not or cannot be measured, then only the available output data can be fitted to a time domain autoregressive moving-average (ARMA) model. This estimation process always produces a minimum phase system. This means that only the natural frequency and damping ratio of the system can be identified. The mode shape cannot be determined uniquely. A new approach has been introduced in this paper to overcome these problems when a structure is randomly excited. The selection of the sampling interval for estimating the modal parameters from a randomly excited structure subjected to unmeasurable inputs is also considered. The theoretical basis of the procedure is presented together with simulation results.

I. Introduction

EXPERIMENTAL modal analysis has become an increasingly important engineering tool during the past 40 years in the aerospace, automotive, and machine tool industries. Modal parameters estimates obtained from experimental modal analysis are being used in the direct solution of vibration and/or acoustic problems, for correlation with output from finite-element programs, and for prediction of changes in system dynamics due to structural changes. In all cases, the quality of the modal parameter estimates is of major concern.

Time series methods have been applied to the synthesis of structural systems excited by random forcing functions as well as to the identification of the natural frequencies and the damping ratio. Autoregressive moving-average (ARMA) models have been used to estimate the characteristics of buildings being excited by wind force and the characteristics of the cutting process that participated with random cutting forces.[1,2] Recently, there has been a great deal of interest in determining modal parameters from measured response data taken on operating systems (e.g., turbulent flow over an airfoil, road inputs to automobiles, and environmental inputs to proposed large space structures). When random input data are not or cannot be measured, then only the available output data can be fitted to a time domain ARMA model. This estimation process always produces a minimum-phase system. The transfer functions defined this way by the ARMA model are successful in the estimation of magnitudes of the true transfer functions but do not give the correct phase information[3,4] except when the true system is minimum phase. In other words, the mode shape cannot be determined uniquely.

The objective of this paper is to solve the aforementioned problem and estimate the modal parameters when the input force is an unmeasured white noise sequence. The selection of the sampling interval for estimating the modal parameters is also considered. Emphasis is placed on the optimum design of uniform data sampling intervals when experimental constraints allow only a limited number of discrete time measurements of the output from the continuous system and the parameters of interest are natural frequencies, damping ratio, and time constant of the continuous system.

II. Statement of the Problem

Consider a structural system with a white noise input sequence $a(k)$ and response $x(k)$, $k = 0,1,2\ldots, L$, then the only available set of data $x(k)$ can be modeled as an ARMA(n,m) model of the form[5]

$$\begin{aligned} x(k) - \phi_1 x(k-1) - \cdots - \phi_n x(k-n) \\ = a(k) - \theta_1 a(k-1) - \cdots - \theta_m a(k-m) \end{aligned} \tag{1}$$

where $\phi_1, \sigma_2, \ldots, \phi_n$ are the autoregressive parameters and $\theta_1, \theta_2, \ldots, \theta_m$ are the moving-average parameters.

Applying z transform to each side of Eq. (1), we have

$$X(z) - \phi_1 X(z) z^{-1} - \cdots - \phi_n X(z) z^{-n}$$

$$= A(z) - \theta_1 A(z) z^{-1} - \cdots - \theta_m A(z) z^{-m}$$

$$H(z) \equiv \frac{\theta(z)}{\phi(z)} = \frac{X(z)}{A(z)} = 1 - \sum_{i=1}^{m} \theta_i z^{-i} \Bigg/ 1 - \sum_{i=1}^{n} \phi_i z^{-i} \tag{2}$$

where $H(z)$ is the transfer function, and $A(z)$ and $X(z)$ are z transforms of the sequences $a(k)$ and $x(k)$. By using only the available set of the data $x(k)$, the modeling procedure is based on the minimization of the following function:

$$V = \sum_{k=1}^{L} [a(k)]^2 \tag{3}$$

By using the total square integral formula,[6] the function V can be written as

$$V = \sum_{k=1}^{L} [a(k)]^2 = \frac{1}{2\pi j} \oint |A(z)|^2 \frac{\mathrm{d}z}{z} \tag{4}$$

where the contour of integration is the unit circle in the z plane. By considering Eq. (2), this equation can also be expressed as

$$\begin{aligned} V = \sum_{k=1}^{L} [a(k)]^2 &= \frac{1}{2\pi j} \oint \left| \frac{\phi(z)}{\theta(z)} X(z) \right|^2 \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi j} \oint \left| \frac{\phi(z)}{\theta(z)} \right|^2 |X(z)|^2 \frac{\mathrm{d}z}{z} \end{aligned} \tag{5}$$

Obviously, the function V does not involve any phase information, since it only involves the absolute value of polynomials $\phi(z)$, $\theta(z)$, and $X(z)$; i.e., the polynomials $\phi(z)$ and $\theta(z)$, which satisfy the condition $V =$ minimum, are not uniquely

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determined. Therefore, even if we constrain the estimated $\phi(z)$ polynomial to be of minimum phase nature, which guarantees a stable model, we cannot decide which of the polynomials $\theta(z)$ with the same magnitude characteristics is the one that truly represents the actual physical system. The problem posed here is closely related to the so-called "stochastic realization," i.e., the problem of obtaining a model of a process $x(k)$, given its covariance function or second-order information.[7] In general, there exist many such models; however, with the inversibility condition, the only stable and stably invertible model is the (unique) innovations representation (IR).[8] The inverse model is the whitening filter that produces a white noise process, the innovations $a(k)$, when driven by the observed data. If the process $x(k)$ is stationary, the problem of obtaining the IR essentially reduces to one of spectral factorization.[9]

In conclusion, when input signals are unmeasurable, we cannot estimate unique transfer functions. Therefore, we have to obtain more information such as velocity to constrain our solution. This condition will be discussed in the next section.

III. Mathematical Formulation

Let a randomly excited structural system with n degree of freedom be expressed by

$$\underset{(n\times n)(n\times 1)}{M\ \ddot{X}} + \underset{(n\times n)(n\times 1)}{C\ \dot{X}} + \underset{(n\times n)(n\times 1)}{K\ X} = \underset{(n\times 1)}{f(t)} \tag{6}$$

where M, C, and K represent the mass, damping, and stiffness matrix, respectively, X is a displacement vector, and f is random excitation forces.

By using a new set of variables, q and $\dot{q}$, Eq. (6) can be rewritten as

$$\underset{(2n\times 1)}{\dot{q}} = \underset{(2n\times 2n)}{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}} \underset{(2n\times 1)}{q} + \underset{(2n\times n)}{\begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}} \underset{(n\times 1)}{f(t)}$$

$$= \underset{(2n\times 2n)(2n\times 1)}{-A\ q} + \underset{(2n\times 1)}{Z(t)} \tag{7}$$

where $I = n\times n$ identity matrix

$$q^T = [x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n]$$

and x_i is the displacement component, $\dot{x}_i$ the velocity component, and q^T the transpose of q.

The matrix equation (7) is the vector first-order differential equation with white noise as the forcing function and is called the continuous vector autoregressive model of first order, i.e., the continuous VAR(1) model. The device of expressing Eq. (6) in the form of Eq. (7) is derived by using the so-called state-space (or Markovian) representation of the relationship between input and output rather than the explicit form.[10] It may be viewed alternatively as a special case of the technique of writing certain types of non-Markov processes involving only finite stage dependence as vector Markov processes.

Although continuous time stochastic processes are frequently encountered in physical science and engineering, with the advent of digital computers, their discretely sampled observations have attracted interests for the purpose of analysis. This naturally leads to the corresponding discrete representation of those processes, i.e., the discrete vector first-order autoregressive models [the discrete VAR(1) models]. The relationship of the continuous VAR(1) and the discrete VAR(1) models to obtain the modal parameters will be discussed in the following subsections.

Discrete VAR(1) Model

The discrete VAR(1) model represents the vector first-order stochastic difference equation. It can be expressed as

$$\underset{(2n\times 1)}{p(k)} = \underset{(2n\times 2n)}{\phi}\ \underset{(2n\times 1)}{p(k-1)} + \underset{(2n\times 1)}{a(k)} \tag{8}$$

where

$$p^T(k) = [x_1(k), x_2(k), \ldots, x_n(k), \dot{x}_1(k), \dot{x}_2(k), \ldots, \dot{x}_n(k)]$$
$$= [p_1(k), p_2(k), \ldots, p_{2n}(k)]$$

and ϕ is the discrete autoregressive matrix, a the discrete vector of random forces, and

$$E\,[a(k)\,a^T(k-l)] = \delta_l \Gamma_a$$

$$\delta_l = \begin{cases} 1 & \text{for} \quad l = 0 \\ 0 & \text{for} \quad l \neq 0 \end{cases}$$

where δ_l is the Kronecker delta function, Γ_a the variance matrix of $a(k)$, and E the expectation operator.

The relationship between input signal a and output signal p is expressed by the Green function matrix G:

$$p(k) = \sum_{j=0}^{\infty} G_j a(k-j)$$
$$= \sum_{j=-\infty}^{k} G_{k-j} a(j) \tag{9}$$

where G_j is a $2n \times 2n$ matrix.

Using Eq. (8) we have

$$p(k) = (I - \phi B)^{-1} a(k)$$
$$= \sum_{j=0}^{\infty} \phi^j a(k-j) \tag{10}$$

where B is a backshift operator and has the property $Bf(k) = f(k-1)$.

From Eqs. (9) and (10), we obtain the Green function matrix of the discrete VAR(1) model:

$$G_j = \phi^j \tag{11}$$

The eigenvalue matrix (Λ) and eigenvector matrix (T) can be solved by

$$(I\lambda - \phi)T = 0 \tag{12}$$

and the matrix ϕ can be expressed by the eigenvalue matrix and the eigenvector matrix:

$$\phi = T\Lambda T^{-1} \tag{13}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{2n} \end{bmatrix}$$

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,2n} \\ T_{2,1} & \cdots & & \\ \vdots & & & \\ T_{2n,1} & \cdots & \cdots & T_{2n,2n} \end{bmatrix} = [T_1, T_2, \ldots, T_{2n}]$$

$$T^{-1} = \begin{bmatrix} T^{1,1} & T^{1,2} & \cdots & T^{1,2n} \\ T^{2,1} & \cdots & & \\ \vdots & & & \\ T^{2n,1} & \cdots & \cdots & T^{2n,2n} \end{bmatrix} = \begin{bmatrix} (T^1)^T \\ (T^2)^T \\ \vdots \\ (T^{2n})^T \end{bmatrix}$$

$$T_i^T = [T_{1,i}, T_{2,i}, \ldots, T_{2n,i}]; \qquad (T^i)^T = [T^{i,1}, T^{i,2}, \ldots, T^{i,2n}]$$

in which the matrix T^{-1} is the inverse matrix of the matrix T. Substituting Eq. (13) into Eq. (11), the following equation can be derived:

$$G_j = \phi^j = T\Lambda^j T^{-1} = \sum_{i=1}^{2n} g_i \lambda_i^j \tag{14}$$

where

$$g_i = T_i (T^i)^T$$

The state covariance matrix function, which indicates how the state variables are affected or related with other state variables at different numbers of lag k, can be expressed as

$$\Gamma_k = E[p(l)p^T(l-k)] \tag{15}$$

Substituting Eq. (10) into Eq. (15), we obtain

$$\Gamma_k = \sum_{i=0}^{\infty} \phi^{i+k}\Gamma_a(\phi^i)^T = \phi^k \sum_{i=0}^{\infty} \phi^i \Gamma_a (\phi^i)^T = \phi^k \Gamma_0 \tag{16}$$

From Eqs. (10) and (14), Eq. (16) can be rewritten as

$$\Gamma_k = \sum_{i=1}^{2n}\left(\sum_{j=1}^{2n} g_i \Gamma_a g_j^T \frac{1}{1-\lambda_i\lambda_j}\right)\lambda_i^k = \sum_{i=1}^{2n} d_i \lambda_i^k \tag{17}$$

where

$$d_i = \sum_{j=1}^{2n} g_i \Gamma_a g_j^T \frac{1}{1-\lambda_i \lambda_j}$$

is a $2n \times 2n$ matrix.

B. Continuous VAR(1) Model

The continuous VAR(1) model represents the vector first-order stochastic differential equation. It can be expressed as

$$\underset{(2n\times 1)}{\dot{q}(t)} + \underset{(2n\times 2n)}{A}\ \underset{(2n\times 1)}{q(t)} = \underset{(2n\times 1)}{Z(t)} \tag{18}$$

where

$$q^T(t) = [x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n]$$
$$= [q_1(t), q_2(t), \ldots, q_{2n}(t)]$$
$$E[Z(t)] = \mathbf{0}$$
$$E[Z(t)Z^T(t-v)] = \delta(v)\Gamma_z$$

where $\delta(v)$ is the Dirac delta function and Γ_z the variance matrix of $Z(t)$.

The Green function matrix $G(t)$ can be expressed as

$$q(t) = \int_0^{\infty} G(\tau) Z(t-\tau)\,\mathrm{d}\tau \tag{19}$$

Solving Eq. (18), we have

$$q(t) = \int_0^{\infty} e^{-A\tau} Z(t-\tau)\,\mathrm{d}\tau \tag{20}$$

From Eqs. (19) and (20), the Green function matrix of the continuous VAR(1) model is

$$G(t) = e^{-At} \tag{21}$$

Equation (21) is commonly called the state transition matrix in linear time-invariant systems.

The eigenvalue matrix (U), eigenvalue ($u_1, u_2, \ldots, u_{2n}$), and the eigenvector matrix ($\mathfrak{Z}$) can be found by solving

$$(Iu + A)\mathfrak{Z} = 0 \tag{22}$$

With the same procedure as that in Sec. III.A, the following equations can be derived:

$$G(t) = \exp(-At) = \exp(\mathfrak{Z} U \mathfrak{Z}^{-1} t)$$
$$= \mathfrak{Z}\exp(Ut)\mathfrak{Z}^{-1} = \sum_{i=1}^{2n} g_i \exp(u_i t) \tag{23}$$

where

$$g_i = \mathfrak{Z}_i (\mathfrak{Z}^i)^T$$

$$U = \begin{bmatrix} u_1 & & & \mathbf{0} \\ & u_2 & & \\ & & \ddots & \\ \mathbf{0} & & & u_{2n} \end{bmatrix}$$

$$\mathfrak{Z} = \begin{bmatrix} \mathfrak{Z}_{1,1} & \mathfrak{Z}_{1,2} & \cdots & \mathfrak{Z}_{1,2n} \\ \mathfrak{Z}_{2,1} & \cdots & & \\ \vdots & & & \\ \mathfrak{Z}_{2n,1} & \cdots & \cdots & \mathfrak{Z}_{2n,2n} \end{bmatrix} = [\mathfrak{Z}_1, \mathfrak{Z}_2, \ldots, \mathfrak{Z}_{2n}]$$

$$\mathfrak{Z}^{-1} = \begin{bmatrix} \mathfrak{Z}^{1,1} & \mathfrak{Z}^{1,2} & \cdots & \mathfrak{Z}^{1,2n} \\ \mathfrak{Z}^{2,1} & \cdots & & \\ \vdots & & & \\ \mathfrak{Z}^{2n,1} & \cdots & \cdots & \mathfrak{Z}^{2n,2n} \end{bmatrix} = \begin{bmatrix} (\mathfrak{Z}^1)^T \\ (\mathfrak{Z}^2)^T \\ \vdots \\ (\mathfrak{Z}^{2n})^T \end{bmatrix}$$

$$\mathfrak{Z}_i^T = [\mathfrak{Z}_{1,i}, \mathfrak{Z}_{2,i}, \ldots, \mathfrak{Z}_{2n,i}]; \qquad (\mathfrak{Z}^i)^T = [\mathfrak{Z}^{i,1}, \mathfrak{Z}^{i,2}, \ldots, \mathfrak{Z}^{i,2n}]$$

Using Eq. (20) the state covariance matrix function of $q(t)$ is given by

$$\Gamma_q(v) = E[q(t+v)q^T(t)]$$
$$= \int_0^{\infty} \exp[-A(t+v)]\Gamma_z \exp[-A(t)]\,\mathrm{d}t$$
$$= \exp(-Av)\Gamma_q(0) \tag{24}$$

From Eqs. (20) and (23), Eq. (24) can be rewritten as

$$\Gamma_q(v) = E[q(t+v)\,q^T(t)]$$
$$= \sum_{i=1}^{2n}\left(\sum_{j=1}^{2n} g_i \Gamma_z g_j^T \frac{-1}{u_i + u_j}\right) e^{u_i v}$$
$$= \sum_{i=1}^{2n} d_i\, e^{u_i v} \tag{25}$$

where

$$d_i = \sum_{j=1}^{2n} g_i \Gamma_z g_j^T \frac{-1}{u_i + u_j}$$

C. Step-by-Step Procedure of the New Method

The way to find the values of the continuous parameters corresponding to the discrete parameters is through the covariance invariant principle[4]; i.e., Γ_k in Eqs. (16) and (17) must be equal to the continuous covariance function $\Gamma(k\Delta)$ in Eqs. (24) and (25) by uniformly sampling. That is,

$$\Gamma_k = \Gamma_q(k\Delta) = \phi^k \Gamma_0 = \sum_{i=1}^{2n} d_i \lambda_i^k$$
$$= e^{-Ak\Delta}\Gamma_q(0) = \sum_{i=1}^{2n} d_i e^{u_i k\Delta} \tag{26}$$

With such treatment we have

$$\lambda_i = e^{u_i\Delta} \tag{27}$$

$$\Gamma_0 = \Gamma_q(0) \tag{28}$$

$$\phi = T\Lambda T^{-1} = \mathfrak{I}\, e^{U\Delta}\mathfrak{I}^{-1} = e^{-A\Delta} \tag{29}$$

and from Eqs. (27) and (29) we obtain

$$T = \mathfrak{I} \tag{30}$$

The modal parameters (i.e., natural frequency, damping ratio, and mode shape) can be obtained by the following procedure:

1) The measured displacement and velocity are fitted into the discrete VAR(1) model. The matrix ϕ in Eq. (8) could be obtained.

2) By solving the eigenvalue-eigenvector problem in Eq. (12), we could get the modal vectors (i.e., eigenvectors) and discrete eigenvalues.

3) Continuous eigenvalues u_i and the A matrix could be found by using Eqs. (27) and (29); i.e., the stochastic model in Eq. (7) could be obtained.

4) The damping ratio ξ_i and natural frequency ω_i are solved by the following equation[11]:

$$u_i, \qquad u_i^* = \omega_i\left(-\xi_i \pm \sqrt{1-\xi_i^2 j}\right) \tag{31}$$

where u_i^* is the complex conjugate of u_i.

The uniqueness representation and parameter estimation of the discrete VAR(1) model are presented in Appendices A and B.

IV. Fisher Information Matrix with Respect to the Discrete Eigenvalues

In this section we shall develop the Fisher information matrix for a VAR(1) model and derive the variance-covariance matrix with respect to the parameters of interest.

Equation (8) can also be expressed as

$$\left(I - \sum_{i=1}^{2n} g_i\lambda_i B\right) x(k) = a(k) \tag{32}$$

If the number of the observations N is large, then the likelihood function[12] can be approximately expressed as

$$f(\beta\,|\,\alpha) = [2\pi \det(\gamma_a)]^{-N/2} \exp\left\{-\sum_{k=1}^{N}[\alpha^T(k)\gamma_a^{-1}\alpha(k)]/2\right\}$$

where

$$\alpha^T(k) = [a_{n+1}(k), a_{n+2}(k), \ldots, a_{2n}(k)]$$

$$E[\alpha(k)\alpha^T(k-l)] = \delta_l\gamma_a$$

$$\beta^T = [\lambda_1, \lambda_2, \ldots, \lambda_{2n}]$$

and $\det(\gamma_a)$ is the determinant of matrix γ_a. The log-likelihood function is

$$L = -(N/2)\log[2\pi\det(\gamma_a)] - \sum_{k=1}^{N}\left\{[\alpha^T(k)\,\gamma_a^{-1}\alpha(k)]/2\right\} \tag{33}$$

Then the $(2n \times 2n)$ matrix $I(\beta)$, which is called the Fisher information matrix for the parameter β, is given by[13]

$$I_{ij}(\beta) = E\left(-\frac{\partial^2 L}{\partial\lambda_i\,\partial\lambda_j}\right) \tag{34}$$

For structural systems, however, the eigenvalue set always contains complex conjugates. Since $\partial L/\partial\lambda_j$ is complex when λ_j is complex, Eq. (34) can be generalized as

$$I_{ij}(\beta) = E\left(-\frac{\partial^2 L}{\partial\lambda_i\,\partial\lambda_j^*}\right) \tag{35}$$

and λ_j^* is the complex conjugate of the λ_j.

From Eqs. (33) and (35), we obtain the (i,j) element of the Fisher information matrix $I(\beta)$ as

$$\begin{aligned} I_{ij}(\beta) &= -E\left(\frac{\partial^2 L}{\partial\lambda_i\,\partial\lambda_j^*}\right) = \frac{1}{2}\sum_{k=1}^{N} E\left\{\frac{\partial^2[\alpha^T(k)\gamma_a^{-1}(k)\alpha]}{\partial\lambda_i\,\partial\lambda_j^*}\right\} \\ &= \frac{1}{2}\sum_{k=1}^{N}\text{trace}\left(E\left\{\frac{\partial^2[\alpha(k)\alpha^T(k)]}{\partial\lambda_i\partial\lambda_j^*}\right\}\gamma_a^{-1}\right) \\ &= \frac{1}{2}\sum_{k=1}^{N}\text{trace}\left[\text{sub}\left(E\left\{\frac{\partial^2[\boldsymbol{a}(k)\boldsymbol{a}^T(k)]}{\partial\lambda_i\,\partial\lambda_j^*}\right\}\right)\gamma_a^{-1}\right] \end{aligned} \tag{36}$$

where the relationship

$$\begin{aligned} \boldsymbol{a}^T(k) &= [0, 0, \ldots, a_{n+1}(k), a_{n+2}(k), \ldots, a_{2n}(k)] \\ &= [0, 0, \ldots, \alpha^T(k)] \end{aligned}$$

was used and the notation sub(...) denotes the lower-right corner $n \times n$ submatrix of the matrix (...). The derivative of $\boldsymbol{a}(k)$ can be found from Eq. (32) as

$$\begin{aligned} \frac{\partial\boldsymbol{a}(k)}{\partial\lambda_i} &= \frac{\partial\left[\left(I - \sum_{l=1}^{2n} g_l\lambda_l B\right)x(k)\right]}{\partial\lambda_i} \\ &= -g_i Bx(k) = -g_i\frac{I-\phi B}{I-\phi B}x(k-1) \\ &= -g_i[I + (\phi B) + (\phi B)^2 + (\phi B)^3 + \ldots]\boldsymbol{a}(k-1) \\ &= -g_i[\boldsymbol{a}(k-1) + \phi\boldsymbol{a}(k-2) + \phi^2\boldsymbol{a}(k-3) + \ldots] \end{aligned} \tag{37}$$

Similarly,

$$\begin{aligned} \frac{\partial\boldsymbol{a}^T(k)}{\partial\lambda_j^*} &= -[\boldsymbol{a}^T(k-1) + \boldsymbol{a}^T(k-2)\phi^{*T} \\ &\quad + \boldsymbol{a}(k-3)(\phi^{*T})^2 + \cdots](\boldsymbol{g}_j^*)^T \end{aligned} \tag{38}$$

Because

$$E[\boldsymbol{a}(k-l)\boldsymbol{a}^T(k-m)] = \begin{cases} 0 & \text{when } l \neq m \\ \Gamma_a & \text{when } l = m \end{cases}$$

and from Eqs. (37) and (38), we obtain

$$\begin{aligned} E&\left\{\sum_{k=1}^{N}\frac{\partial^2[\boldsymbol{a}(k)\boldsymbol{a}^T(k)]}{\partial\lambda_i\,\partial\lambda_j^*}\right\} \\ &= 2\sum_{k=1}^{N} g_i[\Gamma_a + \phi\Gamma_a\phi^{*T} + \phi^2\Gamma_a(\phi^{*T})^2 + \ldots](\boldsymbol{g}_j^*)^T \\ &= 2Ng_i\left[\gamma_a + \sum_{l=1}^{2n}\sum_{m=1}^{2n}\lambda_l\lambda_m^* g_l\Gamma_a(g_m^*)^T + \sum_{l=1}^{2n}\sum_{m=1}^{2n} \right. \\ &\quad \left. \times\, \lambda_l^2(\lambda_m^*)^2 g_l\Gamma_a(g_m^*)^T + \ldots\right](\boldsymbol{g}_j^*)^T \\ &= 2Ng_i\left[\sum_{l=1}^{2n}\sum_{m=1}^{2n}\frac{1}{1-\lambda_l\lambda_m^*}g_l\Gamma_a(g_m^*)^T\right](\boldsymbol{g}_j^*)^T \end{aligned} \tag{39}$$

where

$$\phi^i = \sum_{l=1}^{2n} g_l\lambda_l^i$$

Substituting Eq. (39) into Eq. (36), the (i,j) element of the Fisher information matrix $[I_{ij}(\beta)]$ can be obtained as

$$I_{ij}(\beta) = N\,\mathrm{trace}\left(\mathrm{sub}\left\{g_i\left[\sum_{l=1}^{2n}\sum_{m=1}^{2n}\frac{1}{1-\lambda_l\lambda_m^*}g_l\Gamma_a(g_m^*)^T\right]\times(g_j^*)^T\right\}\gamma_a^{-1}\right) = N\,\mathrm{trace}\left\{\mathrm{sub}[(g)_i R(g^*)_j^T]\gamma_a^{-1}\right\} \quad (40)$$

where

$$R = \sum_{l=1}^{2n}\sum_{m=1}^{2n}\frac{1}{1-\lambda_l\lambda_m^*}(g)_l\Gamma_a(g^*)_m^T$$

V. Fisher Information Matrix with Respect to Natural Frequencies and Damping Ratios

In modal analysis application, the parameters of interest are natural frequencies and damping ratios. From Eqs. (27) and (31) we can find that the transformation is

$$\lambda_k, \lambda_k^* = \exp[-(\xi_k \pm \sqrt{1-\xi_k^2}\,j)\,\omega_k\Delta] \quad (41)$$

The Fisher information matrix for the transformed parameters can be obtained by using the transformation (41) and Eq. (40). Define Y^T as

$$Y^T = [Y_1, Y_2, \ldots, Y_{2n}] = [\omega_1, \xi_1, \omega_2, \xi_2, \ldots, \omega_n, \xi_n]$$

Then, by the chain rule,

$$\frac{\partial a_{n+j}(k)}{\partial Y_i} = \sum_{l=1}^{2n}\frac{\partial a_{n+j}(k)}{\partial\lambda_l}\frac{\partial\lambda_l}{\partial Y_i}\quad (j=1,2,\ldots,n) \quad (42)$$

By taking the expectation of

$$\frac{\partial^2 a_{n+j}(k)}{\partial Y_i\,\partial Y_j^*}$$

the Fisher information matrix for Y is given by

$$I(Y) = Ja^T I(\beta) Ja^* \quad (43)$$

where Ja is the Jacobian matrix given by

$$Ja = \begin{bmatrix} \frac{\partial\lambda_1}{\partial Y_1} & \frac{\partial\lambda_1}{\partial Y_2} & \cdots & \frac{\partial\lambda_1}{\partial Y_{2n}} \\ \frac{\partial\lambda_2}{\partial Y_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial\lambda_{2n}}{\partial Y_1} & \cdots & & \frac{\partial\lambda_{2n}}{\partial Y_{2n}} \end{bmatrix} \quad (44)$$

The elements of Ja can be found using Eq. (41), and the analytical derivatives are

$$\frac{\partial\lambda_i}{\partial\omega_j} = \begin{cases} -(\xi_i + \sqrt{1-\xi_i^2}\,j)\Delta\lambda_i & \text{for } i=j \\ 0 & \text{for } i\neq j \end{cases}$$

$$\frac{\partial\lambda_i}{\partial\xi_j} = \begin{cases} -[1-(\xi_i/\sqrt{1-\xi_i^2}\,j)\omega_i\Delta\lambda_i & \text{for } i=j \\ 0 & \text{for } i\neq j \end{cases}$$

Thus, the Fisher information matrix with respect to ξ_n and ω_n can be found, and the variance-covariance matrix $\mathrm{var}(Y)$ for the transformed parameters can be obtained by inverting the Fisher information matrix, i.e.,

$$\mathrm{var}(Y) = I^{-1}(Y) \quad (45)$$

If the variance-covariance matrix is solved, the standard errors for these parameters can then be obtained from the diagonal elements. The approximate 95% confidence limits of the parameters are[10]

$$Y_i \pm 1.96\sqrt{\mathrm{var}(Y_i)}$$

where $\sqrt{[\mathrm{var}(Y_i)]}$ is the standard error obtained from the diagonal elements of the estimated variance-covariance matrix of Y.

Table 1 Simulation results together with theoretical modal parameters (eigenvalues, natural frequencies, and damping ratios)

Sampling interval, s	Eigenvalue	Damping ratio	Natural frequency, rad/s
1st Mode			
Theoretical	$-1.2142E-2 \pm 0.6170E\ 0j$	$1.9675E-2$	$0.6171E\ 0$
0.03	$-0.6740E-2 \pm 0.6218E\ 0j$	$1.0839E-2$	$0.6218E\ 0$
0.243	$-0.4703E-2 \pm 0.6188E\ 0j$	$0.7600E-2$	$0.6188E\ 0$
2nd Mode			
Theoretical	$-5.3963E-2 \pm 1.6045E\ 0j$	$3.3613E-2$	$1.6054E\ 0$
0.03	$-7.0018E-2 \pm 1.6927E\ 0j$	$4.1330E-2$	$1.6942E\ 0$
0.243	$-7.6611E-2 \pm 1.6052E\ 0j$	$4.7673E-2$	$1.6070E\ 0$
3rd Mode			
Theoretical	$-1.2068E-1 \pm 3.0128E\ 0j$	$4.0023E-2$	$3.0152E\ 0$
0.03	$-2.0303E-1 \pm 2.8950E\ 0j$	$6.9957E-2$	$2.9022E\ 0$
0.243	$-1.3027E-1 \pm 2.9829E\ 0j$	$4.3632E-2$	$2.9857E\ 0$
4th Mode			
Theoretical	$-1.5268E-1 \pm 6.6777E\ 0j$	$2.2858E-2$	$6.6795E\ 0$
0.03	$-2.4305E-1 \pm 6.5901E\ 0j$	$3.6855E-2$	$6.5946E\ 0$
0.243	$-1.2056E-1 \pm 6.6843E\ 0j$	$1.8034E-2$	$6.6853E\ 0$
5th Mode			
Theoretical	$-3.2326E-1 \pm 9.4955E\ 0j$	$3.4298E-2$	$9.4251E\ 0$
0.03	$-4.3795E-1 \pm 9.4569E\ 0j$	$4.6260E-2$	$9.4671E\ 0$
0.243	$-3.4946E-1 \pm 9.4395E\ 0j$	$3.6963E-2$	$9.4460E\ 0$
6th Mode			
Theoretical	$-2.4420E-1 \pm 1.1372E\ 1j$	$2.1468E-2$	$1.1375E\ 01$
0.03	$-2.6060E-1 \pm 1.1326E\ 1j$	$2.3003E-2$	$1.1329E\ 01$
0.243	$-2.3960E-1 \pm 1.1400E\ 1j$	$2.1013E-2$	$1.1402E\ 01$

VI. Choice of Optimal Sampling Interval

In principle, we can differentiate the particular elements of Eq. (45) with respect to the sampling interval and equate the result to 0 to obtain the optimal sampling interval that minimizes the variance of the interested parameter. However, because of the complexity in the nonlinear algebraic equations involved, in general, there is no closed-form solution except for first-order systems (see Appendix C). Consequently, one is compelled to obtain the solution numerically, and it is sometimes the case that a single sampling interval cannot cover the whole frequency range of interest. Therefore, for higher-order systems, the optimal sampling intervals might be respectively chosen for each mode in the system; i.e., one mode has priority at a time. The computational procedures are almost the same as those of first-order systems.

The other alternative to the problem is to choose the optimal sampling interval for ξ and ω by reducing it to one parameter problem. The transformation is

$$q = w_1\omega_1 + w_2\xi_1 + w_3\omega_2 + w_4\xi_2 + \cdots + w_{2n-1}\omega_n + w_{2n}\xi_n = w^TY \tag{46}$$

where w_{2i-1}, $w_{2i}(i=1, 2, ..., n)$ are predetermined weighting factors based on the practical justification about relative importance of w_i and ξ_i $(i=1, 2 ..., n)$. The optimal sampling interval can then be defined as the sampling interval that yields minimum variance of q:

$$\text{var}(q) = w^T\text{var}(Y)w \tag{47}$$

In practical implementation, the exact values of the natural frequencies and damping ratios cannot be known a priori for obtaining the optimum sampling interval. An iterative procedure is suggested by first choosing a reasonable initial guess of sampling interval based on physical reasoning and finding the estimated modal parameters by the procedures presented in Sec. III.C, then acquiring the corresponding Fisher information matrix with respect to those estimated $\hat{\omega}$ and $\hat{\xi}$ by using Eq. (43) and obtaining the var(q) by Eqs. (46) and (47). In consequence, a new sampling interval can be obtained by minimizing the var(q) to refine the estimation accuracy.

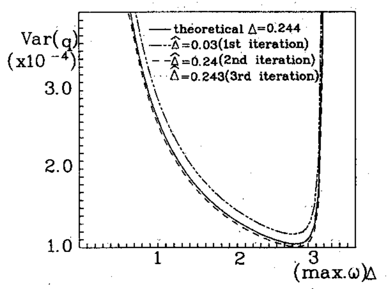

Fig. 1 Typical plot of var(q) vs (maxω)Δ for case 1.

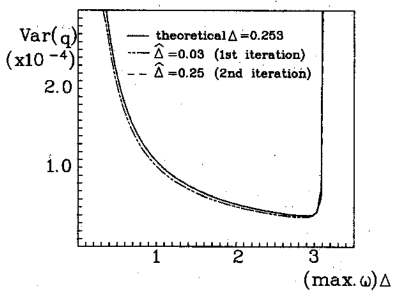

Fig. 2 Typical plot of var(q) vs (maxω)Δ for case 2.

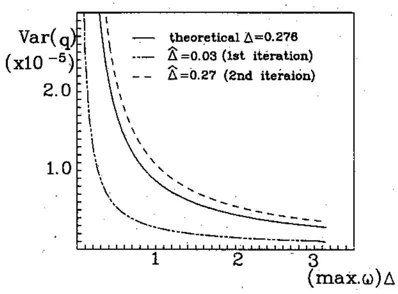

Fig. 3 Typical plot of var(q) vs (maxω)Δ for case 3.

VII. Numerical Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.8 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \\ \ddot{x}_4(t) \\ \ddot{x}_5(t) \\ \ddot{x}_6(t) \end{bmatrix} + \begin{bmatrix} 0.4 & -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0.5 & -0.3 & 0 & 0 & 0 \\ 0 & -0.3 & 0.6 & -0.3 & 0 & 0 \\ 0 & 0 & -0.3 & 0.8 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0.6 & -0.1 \\ 0 & 0 & 0 & 0 & -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \\ \dot{x}_6(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 43 & -40 & 0 & 0 & 0 & 0 \\ -40 & 54 & -14 & 0 & 0 & 0 \\ 0 & -14 & 144 & -130 & 0 & 0 \\ 0 & 0 & -130 & 210 & -80 & 0 \\ 0 & 0 & 0 & -80 & 84 & -4 \\ 0 & 0 & 0 & 0 & 4 & 6.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \\ f_5(t) \\ f_6(t) \end{bmatrix} \tag{48}$$

The lumped-mass system described by Eq. (48) with linear springs and dampers was chosen to illustrate the applicability of this new method. The numerical example was executed on a VAX 8800 digital computer, and a computer package, Matrix$_X$,[14] was used to simulate the system. The band-limited white noise was given to the system as excitation force to generate response. The number of observations was chosen to be 1000 for each variable. Although it is not possible to build a generator of perfect white noise, as long as the flat range of a practical generator extends beyond the frequency response of the system being considered, the "nonwhiteness" will not present any difficulty. The variance vector of exciting random

Table 2 Simulation results together with theoretical modal parameters (modal vectors)

Sampling interval, s	Modal vector		
1st Mode			
Theoretical	(1.0,	1.07E 0−2.51E−3j,	1.22E 0−6.96E−3j
	12.3E 0−7.32E−3j,	1.23E 0−7.15E−3j,	9.07E−1+4.64E−3j)
0.03	(1.0,	1.06E 0−1.43E−3j,	1.25E 0−1.40E−2j
	1.26E 0−1.30E−2j,	1.26E 0−1.17E−2j,	9.12E−1+4.55E−2j)
0.243	(1.0,	1.07E 0−3.27E−3j,	1.22E 0−8.40E−3j
	1.23E 0−8.92E−3j,	1.23E 0−8.46E−3j,	9.19E−1+8.25E−3j)
2nd Mode			
Theoretical	(1.0,	1.01E 0−3.61E−3j,	8.54E−1−6.36E−3j
	8.04E−1−3.74E−3j,	6.44E−1+4.81E−3j,	−3.59E 0+9.98E−2j)
0.03	(1.0,	1.01E 0+2.59E−3j,	7.50E−1+1.26E−2j
	6.96E−1+3.64E−2j,	5.52E−1+6.80E−2j,	−3.07E 0+8.10E−1j)
0.243	(1.0,	1.01E 0+1.80E−3j,	8.41E 0−1.65E−2j
	7.85E−1−1.32E−2j,	6.30E−1−5.19E−3j,	−3.55E 0+4.27E−2j)
3rd Mode			
Theoretical	(1.0,	8.48E−1+8.21E−4j,	−1.38E−1−1.04E−2j
	−2.24E−1−5.47E−3j,	−2.89E−1−1.91E−3j,	6.09E−2−5.49E−4j)
0.03	(1.0,	8.30E−1−3.08E−2j,	−1.59E−1−8.86E−2j
	−2.49E−1−6.84E−2j,	−3.47E−1−4.75E−2j,	2.21E−1−5.59E−2j)
0.243	(1.0,	8.52E−1−1.51E−4j,	−1.35E−1−4.00E−3j
	−2.23E−1+9.13E−3j,	−2.90E−1+1.27E−2j,	3.11E−2+1.28E−2j)
4th Mode			
Theoretical	(1.0,	−4.02E−2−1.72E−2j,	−2.85E 0−3.21E−1j
	−1.19E 0−1.54E−1j,	3.48E 0+4.37E−1j,	−1.20E−1+1.18E−3j)
0.03	(1.0,	−5.45E−2−7.89E−2j,	−2.74E 0−1.08E 0j
	−1.12E 0−6.23E−1j,	3.27E 0+1.26E 0j,	−5.60E−1−1.97E−1j)
0.243	(1.0,	−2.36E−2−1.62E−2j,	−2.70E 0−3.25E−1j
	−1.17E 0−1.61E−1j,	3.27E 0+3.83E−1j,	−3.24E−2+8.15E−2j)
5th Mode			
Theoretical	(1.0,	−1.15E 0+4.09E−3j,	−7.98E−3+9.90E−3j
	1.25E−1−2.56E−2j,	−7.24E−2+1.41E−2j,	1.16E−3−4.45E−4j)
0.03	(1.0,	−1.17E 0−7.07E−2j,	1.06E−1−2.30E−2j
	1.27E−1+3.85E−3j,	−1.32E−1+4.20E−2j,	8.91E−2+4.20E−2j)
0.243	(1.0,	−1.16E 0−1.95E−2j,	6.97E−3−3.27E−3j
	1.30E−1−2.13E−2j,	−6.99E−2−2.19E−2j,	−3.58E−3+4.83E−3j)
6th Mode			
Theoretical	(1.0,	−2.16E 0−9.76E−2j,	8.14E 0+2.71E 0j
	−6.98E 0−2.35E 0j,	2.36E 0+6.88E−1j,	−2.86E−2−1.10E−3j)
0.03	(1.0,	−1.54E 0−1.83E−1j,	6.04E 0+2.39E 0j
	−5.04E 0−2.24E 0j,	1.46E 0+6.77E−1j,	2.69E−1+1.44E−1j)
0.243	(1.0,	−2.37E 0−5.88E−2j,	10.43E 0+3.29E 0j
	−9.05E 0−2.87E 0j,	3.09E 0+7.87E−1j,	−3.57E−3+3.98E−2j)

force was $[1, 1, 0.25, 0.11, 0.16, 0.13]^T$, and the sampling interval was 0.03 s. From the results shown in Tables 1 and 2, the estimated modal parameters are quite close to the theoretical ones. The reasonableness of the proposed method was demonstrated. For the optimal sampling interval, three cases are considered: 1) the sixth mode is of interest; i.e., the weighting vector w is $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.5, 0.5]^T$. 2) The even weighting of the six natural frequencies is considered; i.e., the weighting vector w is $[0.167, 0, 0.167, 0, 0.167, 0, 0.167, 0, 0.167, 0, 0.167, 0]^T$. 3) The even weighting of the six damping ratios is considered; i.e., the weighting vector w is $[0, 0.167, 0, 0.167, 0, 0.167, 0, 0.167, 0, 0.167, 0, 0.167]^T$. The initial guess of sampling interval was chosen as 0.03 s for those three cases. In case 1, the final sampling interval was obtained to be 0.243 s after three iterations. The results are shown in Tables 1 and 2, which indicate that the estimation accuracy of the sixth mode is improved. The variance of q vs the product of sampling interval and the highest natural frequency of the system is shown in Fig. 1 for each iteration. For cases 2 and 3, the typical plots are shown in Figs. 2 and 3. The numerical scheme converged rapidly after few iterations, and the final results are 0.25 and 0.27 s for cases 2 and 3, respectively.

VIII. Concluding Remarks

This paper presented an approach to overcome the difficulty of nonuniqueness of mode shape in modal analysis when random input data are not or cannot be measured. Also, a work regarding the selection of optimal sampling interval is discussed. The additional velocity information that in consequence can be used in setting up the state covariance matrix at the points of measurement is the key point of this approach. The selection of the optimal sampling interval is based on the fact that the length of the confidence interval (or variance) of an estimated parameter is a measure of the accuracy of estimation; hence, for a vector of parameters, the optimization criterion is to minimize the particular elements of the covariance matrix of the estimated continuous parameter vector as a function of the sampling interval. One drawback to the method is that the full states must be measured in order to back out the state transition matrix. This cannot be accomplished in practice, and the method should be modified to include unmeasured states.

Appendix A: Uniqueness of the Discrete VAR(1) Model

Equation (8) can be rewritten as

$$p_i(k) = \sum_{j=1}^{2n} \phi_{ij} p_j(k-1) + a_i(k) \qquad (i = 1, 2, \ldots, 2n) \tag{A1}$$

$$[p_i(k) - \phi_{ii} p_i(k-1)] = \sum_{\substack{j=1 \\ j \neq i}}^{2n} \phi_{ij} p_j(k-1) + a_i(k) \qquad (i = 1, 2, \ldots, 2n) \tag{A2}$$

Applying the z transform to each side of Eq. (A2),

$$A_i(z) = \phi_{ii}(z)p_i(z) - \sum_{\substack{j=1 \\ j \neq i}}^{2n} \phi_{ij}(z)p_j(z) \qquad (i = 1, 2, \ldots, 2n) \tag{A3}$$

where

$$\phi_{ii}(z) = 1 - \phi_{ii}z^{-1}$$

$$\phi_{ij}(z) = \phi_{ij}z^{-1} \qquad \text{for} \qquad i \neq j$$

and $A_i(z)$ and $p_i(z)$ are the z transform of $a_i(k)$ and $p_i(k)$.

The function V_i can be expressed as

$$V_i = \sum_{k=1}^{L} [a_i(k)]^2 = \frac{1}{2\pi j} \oint \left| \phi_{ii}(z)p_i(z) - \sum_{\substack{j=1 \\ j \neq i}}^{2n} \phi_{ij}(z)p_j(z) \right|^2 \frac{dz}{z}$$

$$= \frac{1}{2\pi j} \oint \Bigg[\left| \sum_{j=1}^{2n} \phi_{ij}(z)p_j(z) \right|^2 + \sum_{\substack{k=1 \\ k \neq i}}^{2n} \sum_{\substack{j=1 \\ j \neq i,k}}^{2n} \phi_{ij}(z)p_j(z)\phi_{ik}$$

$$(-z)p_k(-z) - \phi_{ii}(z)p_i(z) \sum_{\substack{j=1 \\ j \neq i}}^{2n} \phi_{ij}(-z)p_j(-z)$$

$$- \phi_{ii}(-z)p_j(-z) \sum_{\substack{j=1 \\ j \neq i}}^{2n} \phi_{ij}(z)p_j(z) \Bigg] \frac{dz}{z}$$

$$(i = 1, 2, \ldots, 2n) \tag{A4}$$

From this relation we see that the phase of polynomials ϕ_{ij} $(i,j = 1, 2, \ldots, 2n)$ are involved. This fact implies that the modal parameters of the discrete VAR(1) can be determined uniquely.

Appendix B: Parameter Estimation of the Discrete VAR(1) Model

To estimate the parameter matrix ϕ, the sum of the cost function V_i $(i = 1, 2, \ldots, 2n)$ must be minimized.

Taking transpose to each side of Eq. (8), we obtain

$$p^T(k) = p^T(k-1)\,\phi^T + a^T(k) \tag{B1}$$

Now suppose a set of $L + 1$ measurements is obtained from the system response at times $t_0, t_1, \ldots, t_L$. We can then arrange the equations in the following form:

$$C = D\phi^T + \epsilon \tag{B2}$$

where

$$C = \begin{bmatrix} p^T(1) \\ p^T(2) \\ \vdots \\ p^T(L) \end{bmatrix}, \quad D = \begin{bmatrix} p^T(0) \\ p^T(1) \\ \vdots \\ p^T(L-1) \end{bmatrix}, \quad \text{and} \quad \epsilon = \begin{bmatrix} a^T(1) \\ a^T(2) \\ \vdots \\ a^T(L) \end{bmatrix}$$

The sum of the cost function can be written as

$$V = \sum_{i=1}^{2n} \sum_{k=1}^{L} a_i^2(k) = \text{trace}\,[\epsilon^T\epsilon]$$

$$= \text{trace}\,[(C - \phi D^T)^T(C - \phi D^T)]$$

$$= \text{trace}\,[C^TC - \phi D^TC - C^TD\phi^T + \phi D^TD\phi^T] \tag{B3}$$

Differentiate V with respect to the parameter matrix ϕ and equate the result to a zero matrix to determine the condition of the estimate $\hat{\phi}$ that minimizes V. Thus, we obtain

$$\left. \frac{\partial V}{\partial \phi} \right|_{\phi = \hat{\phi}} = 0 - C^TD - C^TD + 2\phi D^TD = 0 \tag{B4}$$

where the following relations are used:

$$\frac{\partial(\text{trace}[ABA^T])}{\partial A} = 2AB$$

where B is symmetric.

$$\frac{\partial(\text{trace}[AB])}{\partial A} = B^T$$

Then the parameter matrix ϕ can be estimated in the following form:

$$\hat{\phi} = (C^TD)(D^TD)^{-1} \tag{B5}$$

It is noted that the parameter matrix estimated by the preceding method is a kind of linear least-squares method that is equivalent to the maximum likelihood approach under an imposed random input condition.

Appendix C: First-Order Processes

The first-order process

$$(D + \alpha)X = Z \tag{C1}$$

has the discrete representation

$$X(k) = \phi X(k-1) + a(k) \tag{C2}$$

where

$$\phi = e^{-\alpha\Delta} = e^{-\Delta/\tau} = \lambda \tag{C3}$$

and $X(k)$, ϕ, and $a(k)$ are scalars.

From Eqs. (40) and (B3), the 1×1 Fisher information matrix for λ is

$$I(\lambda) = \frac{N}{1 - \phi^2} \tag{C4}$$

From Eq. (44), the Jacobian matrix is

$$Ja = \frac{\partial \phi}{\partial \tau} = \frac{\Delta \phi}{\tau^2} \tag{C5}$$

Therefore, the Fisher information matrix for τ is

$$I(\tau) = \frac{N(\Delta\phi)^2}{\tau^4(1 - \phi^2)} \tag{C6}$$

According to Eq. (45), the variance of τ is

$$\text{var}(\tau) = \frac{\tau^4(1 - \phi^2)}{N(\Delta\phi)^2} \tag{C7}$$

The optimal sampling interval (Δ) is the one that yields the smallest var(τ). By taking the partial derivative of var(τ) with respect to Δ and setting it to zero, the optimal Δ can be derived as

$$\frac{\partial \text{var}(\tau)}{\partial \Delta} = \frac{2\tau}{N} \left(\frac{\tau}{\Delta} \right)^3 \left[\left(\frac{\Delta}{\tau} - 1 \right) e^{2\Delta/\tau} + 1 \right] = 0 \tag{C8}$$

When N and τ are constant,

$$\left(\frac{\Delta}{\tau} - 1 \right) e^{2\Delta/\tau} + 1 = 0 \tag{C9}$$

It is concluded that the optimal Δ for a first-order dynamic system is

$$\Delta = 0.7968\tau \tag{C10}$$

and $\phi = e^{-0.7968} = 0.45$ where τ is the time costant of the continuous system. In other words, if the data observed were the realization from the first-order continuous stochastic system, one can consider the sampling interval to be adequate if the value of the estimated parameter ϕ of the discrete model is in the neighborhood of 0.45.

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